Exercise 24

Solve the initial-value problem.

$$4y'' + 4y' + 3y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Plug these formulas into the ODE.

$$4(r^2e^{rx}) + 4(re^{rx}) + 3(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$4r^2 + 4r + 3 = 0$$

Solve for r.

$$r = \frac{-4 \pm \sqrt{16 - 4(4)(3)}}{2(4)} = \frac{-4 \pm \sqrt{-32}}{2(4)} = -\frac{1}{2} \pm \frac{i}{\sqrt{2}}$$

$$r = \left\{-\frac{1}{2} - \frac{i}{\sqrt{2}}, -\frac{1}{2} + \frac{i}{\sqrt{2}}\right\}$$

Two solutions to the ODE are $e^{(-1/2-i/\sqrt{2})x}$ and $e^{(-1/2+i/\sqrt{2})x}$. By the principle of superposition, then,

$$\begin{aligned}
y(x) &= C_1e^{(-1/2-i/\sqrt{2})x} + C_2e^{(-1/2+i/\sqrt{2})x} \\
&= C_1e^{-x/2}e^{-ix/\sqrt{2}} + C_2e^{-x/2}e^{ix/\sqrt{2}} \\
&= e^{-x/2}(C_1e^{-ix/\sqrt{2}} + C_2e^{ix/\sqrt{2}}) \\
&= e^{-x/2}\left[C_1\left(\cos\frac{x}{\sqrt{2}} - i\sin\frac{x}{\sqrt{2}}\right) + C_2\left(\cos\frac{x}{\sqrt{2}} + i\sin\frac{x}{\sqrt{2}}\right)\right] \\
&= e^{-x/2}\left[(C_1 + C_2)\cos\frac{x}{\sqrt{2}} + (-iC_1 + iC_2)\sin\frac{x}{\sqrt{2}}\right] \\
&= e^{-x/2}\left(C_3\cos\frac{x}{\sqrt{2}} + C_4\sin\frac{x}{\sqrt{2}}\right).
\end{aligned}$$

Differentiate the general solution.

$$y'(x) = -\frac{1}{2}e^{-x/2}\left(C_3\cos\frac{x}{\sqrt{2}} + C_4\sin\frac{x}{\sqrt{2}}\right) + e^{-x/2}\left(-\frac{C_3}{\sqrt{2}}\sin\frac{x}{\sqrt{2}} + \frac{C_4}{\sqrt{2}}\cos\frac{x}{\sqrt{2}}\right)$$

Apply the initial conditions to determine C_1 and C_2.

$$y(0) = C_3 = 0$$

$$y'(0) = -\frac{C_3}{2} + \frac{C_4}{\sqrt{2}} = 1$$

Solving this system of equations yields $C_3 = 0$ and $C_4 = \sqrt{2}$. Therefore, the solution to the initial value problem is

$$y(x) = \sqrt{2}e^{-x/2}\sin\frac{x}{\sqrt{2}}.$$

Below is a graph of $y(x)$ versus x.